Extrapolation of Vector valued Rearrangement Operators II

Paul F.X. Müller [1]

February 9, 2009

[1]Supported by the Austrian Science Foundation P 20166-N18,
2000 MSC 46B42, 46B70,47B37.

Abstract

We determine the extrapolation law of rearrangement operators acting on the Haar system in vector valued H^p spaces: If $0 < q \le p < 2$, then,

$$\|T_{\tau,q} \otimes \ \mathrm{Id}_X\|_q^{\frac{q}{2-q}} \le A(p,q)\|T_{\tau,p} \otimes \ \mathrm{Id}_X\|_p^{\frac{p}{2-p}}.$$

For a fixed Banach space X, the extrapolation range $0 < q \le p < 2$ is optimal. If, however, there exists $1 < p_0 < \infty$, so that

$$\|T_{\tau,p_0} \otimes \ \mathrm{Id}_E\|_{L_E^{p_0}} < \infty, \quad \text{for each UMD space E,}$$

then for any $1 < p < \infty$,

$$\|T_{\tau,p} \otimes \ \mathrm{Id}_E\|_{L_E^p} < \infty,$$

for any UMD space E. (The value $p_0 = 2$ is *not* excluded.)

1 Introduction

In this note we identify the extrapolation law of rearrangement operators acting on the Haar system in vector valued H^p spaces: Rearrangement operators are given by an injective map τ acting on dyadic intervals. We have

$$T_{\tau,p}\left(\frac{h_I}{|I|^{1/p}}\right) = \frac{h_{\tau(I)}}{|\tau(I)|^{1/p}}, \quad 0 < p < 2,$$

and by linear extension obtain an operator on the span of the Haar system If $0 < q \leq p < 2$, then, we prove that

$$\|T_{\tau,q} \otimes \ \mathrm{Id}_X\|_q^{\frac{q}{2-q}} \leq A(p,q)\|T_{\tau,p} \otimes \ \mathrm{Id}_X\|_p^{\frac{p}{2-p}}, \tag{1}$$

as well as

$$\|T_{\tau,q} \otimes \ \mathrm{Id}_X\|_q \leq A(p,q)\|T_{\tau,p}\|_p^{2\left(\frac{1}{q}-\frac{1}{p}\right)}\|T_{\tau,p} \otimes \ \mathrm{Id}_X\|_p. \tag{2}$$

By arithmetic (1) is implied by (2). We define in this paper (see Section 2) the norm in vector valued H^p spaces as the L^p norm of Rademacher averages, and use systematically the notation

$$\|T_{\tau,p}\|_p = \|T_{\tau,p} : H^p \to H^p\|,$$

$$\|T_{\tau,p} \otimes \ \mathrm{Id}_X\|_p = \|T_{\tau,p} \otimes \ \mathrm{Id}_X : H^p_X \to H^p_X\|$$

and

$$\|T_{\tau,p} \otimes \ \mathrm{Id}_X\|_{L^p_X} = \|T_{\tau,p} \otimes \ \mathrm{Id}_X : L^p_X \to L^p_X\|.$$

We put the result of this paper, (1) and (2), into perspective by reviewing its predecessors.

Scalar valued extrapolation. The extrapolation law [5] for scalar valued rearrangement operators on dyadic H^p spaces is this,

$$a_{p,q}\|T_{\tau,q}\|_q^{q/(2-q)} \leq \|T_{\tau,p}\|_p^{p/(2-p)} \leq A_{p,q}\|T_{\tau,q}\|_q^{q/(2-q)}, \qquad 0 < q \leq p < 2. \tag{3}$$

Thus, boundedness of $T_{\tau,p}$ on H_p for one value of $0 < p < 2$ implies boundedness of $T_{\tau,q}$ on H_q for all values of q where $0 < q < 2$. Separately, the

boundedness of $T_{\tau,1}$ on H^1 is equivalent to $\sigma = \tau^{-1}$ respecting Carleson Constants. Write

$$[\![\mathcal{C}]\!] = \sup_{I\in\mathcal{C}} \frac{1}{|I|} \sum_{J\subseteq I, J\in\mathcal{C}} |J|,$$

then [11]

$$\|T_{\tau,1}\|_1 \sim \sup \frac{[\![\sigma(\mathcal{C})]\!]}{[\![\mathcal{C}]\!]}, \tag{4}$$

where the supremum is extended over all $\mathcal{C} \subseteq \tau(\mathcal{D})$.

Vector valued rearrangements. By way of example [3] it is easy to see that isolating intrinsic criteria characterizing boundedness of $T_{\tau,p} \otimes \mathrm{Id}_X$, and the search for extrapolation theorems represent two different lines of research, both of which are different from the scalar valued setting.

For a rearrangement τ_0 defined in [3] the boundedness of $T_{\tau_0,p} \otimes \mathrm{Id}_X$, $1 < p \leq 2$, implies Rademacher-type p for X. At the same time the scalar valued extension of $T_{\tau_0,1}$ is bounded on H^1 while its inverse is unbounded. For τ_0 and $1 < p \leq 2$ we have [3],

$$\|T_{\tau_0,1}\|_1 < \infty, \quad \|T^{-1}_{\tau_0,1}\|_1 = \infty \quad \text{and} \quad \mathrm{Type}_p(X) \leq C\|T_{\tau_0,p} \otimes \mathrm{Id}_X\|_{L^p_X}. \tag{5}$$

This example puts restrictions on possible extrapolation theorems for vector valued rearrangement operators. For instance the right hand side estimate in (3) is ruled out when (5) holds. (Just recall how the Rademacher-type of a Banach space depends on p.) In [3] we defined τ_1 such that the scalar valued extension $T_{\tau_1,1}$ is an isomorphism on H^1 and the boundedness of $T_{\tau_1,1} \otimes \mathrm{Id}_X$ on L^p_X implies the UMD condition for X.

We formulate now three general extrapolation estimates that are not yet ruled out by the examples discussed above.

1. The first concerns the extrapolation of isomorphisms across the entire scale of vector valued $L^p (1 < p < \infty)$ spaces. Let X satisfy the UMD property, assume $T_{\tau,1}$ is a scalar valued isomorphism on H^1 and

   $$\|T_{\tau,2} \otimes \mathrm{Id}_X\|_{L^2_X} \|T^{-1}_{\tau,2} \otimes \mathrm{Id}_X\|_{L^2_X} < \infty \quad ,$$

   then

   $$\|T_{\tau,p} \otimes \mathrm{Id}_X\|_{L^p_X} \|T^{-1}_{\tau,p} \otimes \mathrm{Id}_X\|_{L^p_X} < \infty, \quad 1 < p < \infty.$$

   This result is in [3] were the proof is based on geometric and combinatorial characterizations of rearrangements τ when $T_{\tau,1}$ is a scalar valued isomorphism on H^1.

2. The second extrapolation estimate that is not ruled out by the examples discussed above asserts the following. If

$$\|T_{\tau,1}\|_1 < \infty \quad \text{and} \quad \|T_{\tau,2} \otimes \, \mathrm{Id}_X\|_2 < \infty,$$

then

$$\|T_{p,\tau} \otimes \, \mathrm{Id}_X\|_p < \infty, \quad 0 < p \leq 2.$$

Related are the estimates

$$\|T_{\tau,q} \otimes \, \mathrm{Id}_X\|_q^{q/(2-q)} \leq A(p,q)\|T_{\tau,p} \otimes \, \mathrm{Id}_X\|_p^{p/(2-p)}, \quad 0 < q < p < 2. \tag{6}$$

3. The above extrapolation estimates were stated for one fixed Banach space X. The following assumes boundedness of the rearrangement operator for *each* Banach space with the UMD property. Clearly this is a more restrictive hypothesis on the underlying rearrangement, so the resulting extrapolation estimates should be stronger. If there exists $1 < p_0 < \infty$ so that

$$\|T_{\tau,p_0} \otimes \, \mathrm{Id}_E\|_{L_E^{p_0}} < \infty, \quad \text{for each UMD space E,} \tag{7}$$

then for all $1 < p < \infty$

$$\|T_{\tau,p} \otimes \, \mathrm{Id}_E\|_{L_E^p}\|T_{\tau,p}^{-1} \otimes \, \mathrm{Id}_E\|_{L_E^p} < \infty, \tag{8}$$

for each UMD space E. (Note that now $p_0 = 2$ is permissible in the hypothesis (7).)

In this paper we prove that (7) implies (8), provide a proof of (6), and of the implication stated before (6). We point out two direct predecessors to the present work. In [3], we applied Maurey's [10] extrapolation-by-factorization method to $\tau-$monotone operators. For $|I| = |\tau(I)|$ and UMD spaces X we proved that

$$\|T_{\tau,q} \otimes \, \mathrm{Id}_X\|_{L_X^q}^{q/(2-q)} \leq A(p,q,X)\|T_{\tau,p} \otimes \, \mathrm{Id}_X\|_{L_X^p}^{p/(2-p)}, \quad 1 < q \leq p \leq 2.$$

The second predecessor is K. Smela's [14] very recent proof of the scalar extrapolation theorems in [5]. The results of the present paper were obtained by comparing the integral estimates for the maximal functions

$$\mu_{\mathcal{H}}(t) = \sup_{I \in \mathcal{H}} \frac{|\sigma(I)|}{|I|} 1_I(t) \qquad \text{where} \quad \sigma = \tau^{-1}, \tag{9}$$

used by K. Smela [14], to the methods employed in [5, 3] and [11].

Acknowledgement: I would very much like to thank Stefan Geiss for helpful discussions during the preparation of this work and for suggestions improving its presentation.

2 Preliminaries

Here we collect frequently used facts and theorems. We routinely use [12] as reference.

Collections of dyadic Intervals. Let $\mathcal{D}$ denote the collection of (half-open) dyadic intervals contained in the unit interval

$$[(k-1)2^{-n}, k2^{-n}[, \quad 1 \le k \le 2^n, \quad n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ write $\mathcal{D}_n = \{I \in \mathcal{D} : |I| = 2^{-n}\}$. For a collection of dyadic intervals $\mathcal{E}$ we us the $*-$ notation to to denote the pointset covered by $\mathcal{E}$ thus

$$\mathcal{E}^* = \bigcup_{I \in \mathcal{E}} I.$$

Given $I \in \mathcal{D}$ we denote by $G_1(I, \mathcal{E})$ the maximal dyadic intervals that are in $\mathcal{E}$ and strictly contained in I. Note that by telescoping, for a dyadic interval I,

$$|I| = \sum_{K \in \mathcal{E}, K \subseteq I} |K| - |G_1(K, \mathcal{E})^*|.$$

The $n-$th generation of the dyadic intervals in $\mathcal{E}$ underneath K is defined inductively as

$$G_n(K|\mathcal{E}) = \bigcup_{J \in G_{n-1}(K|\mathcal{E})} G_1(J|\mathcal{E}).$$

Let $\mathcal{L}$ be a collection of dyadic intervals. We say that $\mathcal{B}(I) \subseteq \mathcal{L}$ is a block of dyadic intervals in $\mathcal{L}$ if the following conditions hold.

1. In $\mathcal{B}(I)$ has a unique maximal interval, namely the interval I.
2. If $J \in \mathcal{B}(I)$ and $K \in \mathcal{L}$ then

$$J \subseteq K \subseteq I \quad \text{implies} \quad K \in \mathcal{B}(I).$$

The Haar system. Denote by $\{h_I : I \in \mathcal{D}\}$ the $L^\infty-$ normalized Haar system, where h_I is supported on I and

$$h_I = \begin{cases} 1 & \text{on the left half of } I; \\ -1 & \text{on the right half of } I. \end{cases}$$

For $f \in L^p$ we define its dyadic square function as

$$S(f) = \left(\sum_{I \in \mathcal{D}} \langle f, \frac{h_I}{|I|} \rangle^2 1_I \right)^{1/2} .$$

The Marcinkiewicz- Zygmund interpretation of R.E.A.C. Paley's theorem asserts that

$$c_p \|f\|_{L^p} \leq \|S(f)\|_{L^p} \leq C_p \|f\|_{L^p}, \quad (1 < p < \infty).$$

Given $0 < q < \infty$ define dyadic H^q, to be the completion of $\operatorname{span}\{h_I : I \in \mathcal{D}\}$ under the (quasi-) norm is given by

$$\|f\|_{H^q} = \|S(f)\|_{L^q}.$$

The dual of dyadic H^1. Define $f \in \mathrm{BMO}$ if

$$\|f\|^2_{\mathrm{BMO}} = \sup_{I \in \mathcal{D}} \frac{1}{|I|} \| \sum_{J \subseteq I} \langle f, \frac{h_J}{|J|} \rangle h_J \|_2^2 < \infty.$$

Let $\mathcal{L} = \{I \in \mathcal{D} : \langle f, h_I \rangle \neq 0\}$ then

$$\|f\|_2^2 \leq \|f\|^2_{\mathrm{BMO}} |\mathcal{L}^*|.$$

The space BMO is (identified with) the dual to H^1. The pairing between $f \in \mathrm{BMO}$ and $g \in H^1$ is

$$\langle f, g \rangle = \lim_{n \to \infty} \int_0^1 f_n g dt, \quad \text{where} \quad f_n = \sum_{\{I : |I| > 2^{-n}\}} \langle f, \frac{h_I}{|I|} \rangle h_I.$$

Kahane's principle of contraction and Kahane's inequality. See [7], [15]. Let $\{r_n\}$ denote the sequence of independent $\{+1,-1\}$ valued Rademacher functions. Let $x_n \in X$ be a sequence in a Banach space X and let $a_n \in \mathbb{R}$ so that $|a_n| \leq 1$. Then,

$$\int_0^1 \|\sum_{n=1}^N r_n(t)a_n x_n\|_X dt \leq \int_0^1 \|\sum_{n=1}^N r_n(t)x_n\|_X dt.$$

We apply the principle of contraction in combination with the Kahane's inequality asserting that

$$\left(\int_0^1 \|\sum_{n=1}^N r_n(t)x_n\|_X^p dt\right)^{1/p} \leq C_p \int_0^1 \|\sum_{n=1}^N r_n(t)x_n\|_X dt, \quad 1 < p < \infty.$$

Vector valued dyadic Hardy Spaces . See [2, 13]. Given a Banach space X and $x_I \in X$, define $f = (x_I : I \in \mathcal{D})$ to be the X valued vector indexed and ordered by the dyadic intervals. Define the square function of f as

$$\mathbb{S}(f)(t) = \lim_{n\to\infty}\left(\int_0^1 \|\sum_{\{I:|I|\geq 2^{-n}\}} r_I(s)x_I h_I(t)\|_X^2 ds\right)^{1/2},$$

where $\{r_I\}$ is an enumeration of the independent Rademacher system. Let $0 < p < \infty$. We say that $f \in H_X^p$ if

$$\|f\|_{H_X^p} = \|\mathbb{S}(f)\|_{L^p} < \infty.$$

We (should not hesitate to) identify $f = (x_I : I \in \mathcal{D})$ with its formal Haar series $f = \sum_{I\in\mathcal{D}} x_I h_I$. If $1 < p < \infty$ and if X has the UMD property, L_X^p (the Bochner-Lebesgue space) and H_X^p coincide with equivalent norms.

Rearrangement Operators. Let $0 < p < \infty$. Assume $\tau : \mathcal{D} \to \mathcal{D}$ is injective and $f = (x_I : I \in \mathcal{D})$ in H_X^p. The collection of dyadic intervals $\{I \in \mathcal{D} : x_I \neq 0\}$ is the Haar support of f. Define the rearrangement operator $T_{\tau,p} \otimes \mathrm{Id}_X$ in terms of formal Haar series by the relation

$$T_{\tau,p} \otimes \mathrm{Id}_X(f) = \sum_{I\in\mathcal{D}} x_I \left(\frac{|I|}{|\tau(I)|}\right)^{1/p} h_{\tau(I)}.$$

Equivalently, in vector notation,

$$T_{\tau,p} \otimes \ \mathrm{Id}_X(f) = \left(x_{\sigma(J)} \left(\frac{|\sigma(J)|}{|J|} \right)^{1/p} : J \in \tau(\mathcal{D}) \right),$$

where $\sigma = \tau^{-1} : \tau(\mathcal{D}) \to \mathcal{D}$. We write

$$\|T_{\tau,p} \otimes \ \mathrm{Id}_X : H^p_X \to H^p_X\| = \sup \left\{ \|T_{\tau,p} \otimes \ \mathrm{Id}_X(f)\|_{H^p_X} \right\},$$

where the supremum is extended over all f in the unit ball of H^p_X with finite Haar support.

Dyadic Atoms. Let $0 < p < \infty$ and $b_J \in X$. Define $a = (b_J : J \in \mathcal{D})$ to be a dyadic H^p_X- atom if there exists a dyadic interval I so that

$$\operatorname{supp} \mathbb{S}(a) \subseteq I \quad \text{and} \qquad \|\mathbb{S}(a)\|_\infty \le |I|^{-1/p}.$$

Note that $\|a\|_{H^p_X} \le 1$, for a dyadic H^p_X- atom.

Atomic decomposition. Let $0 < p < \infty$ and fix $f = (x_J : J \in \mathcal{D})$ such that $f \in H^p_X$. We employ the atomic decomposition that results from stopping time arguments applied to the square function $\mathbb{S}(f)$: There exists a decomposition of $\mathcal{D}$ into blocks of dyadic intervals $\{\mathcal{B}(I) : I \in \mathcal{E}\}$ and integers $n(I) \in \mathbb{Z}$ so that

$$\sup_{I \in \mathcal{E}} \frac{1}{|I|} \sum_{J \subseteq I, J \in \mathcal{E}} |J| \le 4, \tag{10}$$

$$\mathbb{S}(f_I) \le 2^{n(I)}, \quad \text{for} \quad f_I = \sum_{J \in \mathcal{B}(I)} x_J h_J, \tag{11}$$

and

$$c\|f\|^p_{H^p_X} \le \sum_{I \in \mathcal{E}} |I| 2^{pn(I)} \le A_p \|f\|^p_{H^p_X}. \tag{12}$$

Define

$$\lambda_I = |I|^{1/p} 2^{n(I)}, \quad I \in \mathcal{E}.$$

By (11), $a_I = f_I/\lambda_I$ is a dyadic H^p_X- atom so that

$$f = \sum_{I \in \mathcal{E}} \lambda_I a_I \quad \text{(formal Haar series)}$$

and by (12),

$$c_p\|f\|^p_{H^p_X} \le \sum_{I\in\mathcal{E}} \lambda_I^p \le A_p\|f\|^p_{H^p_X}.$$

The atomic decomposition as cited above originates with [6]. The decomposition of $\mathcal{D}$ into blocks $\mathcal{B}(I)$ is described (for instance) in [12, Pages 42-44]; the right hand side estimate of (12) transfers directly to the range $0 < p < \infty$ and to the square function defining the spaces H^p_X. For the left hand side estimate of (12) we distinguish between the cases $0 < p < 1$ and $1 \le p < \infty$. For $0 < p < 1$ use the quasi-triangle inequality for the spaces H^p_X. For $1 \le p < \infty$ exploit (10) and adapt the proof of [4, Lemma 3] to yield

$$\|f\|^p_{H^p_X} \le C\sum_{I\in\mathcal{E}} \|f_I\|^p_{H^p_X} \quad (1 \le p < \infty),$$

where $C > 0$ depends just on the upper estimate (10) for the Carleson constant of $\mathcal{E}$.

3 Extrapolation by Factorization and Carleson measure

In this section we prove the results of this paper. Let $\tau : \mathcal{D} \to \mathcal{D}$ be injective with inverse $\sigma : \tau(\mathcal{D}) \to \mathcal{D}$. K. Smela based his proof [14] on the fact that the maximal function $\mu_{\mathcal{H}}(t)$ mediates between $T_{\tau,1}$ and $T_{\tau,2}$. This is the link between extrapolation of general rearrangement operators and factorization that extends the use of $\tau-$monotone operators in [3]. We let $f = \sum_{J\in\mathcal{L}} a_J h_J$, with $a_J \in \mathbb{R}$. For $t \in [0,1]$ fixed we estimate square functions

$$\begin{aligned} S(T_{\tau,1}(f))(t) &= \left(\sum_{J\in\mathcal{L}} a_J^2 1_{\tau(J)}(t)\frac{|J|^2}{|\tau(J)|^2}\right)^{1/2} \\ &\le \left(\sup_{J\in\mathcal{L}} \frac{|J|}{|\tau(J)|}1_{\tau(J)}(t)\right)^{1/2}\left(\sum_{J\in\mathcal{L}} a_J^2 1_{\tau(J)}(t)\frac{|J|}{|\tau(J)|}\right)^{1/2}. \end{aligned}$$

The right hand side factor coincides with $S(T_{\tau,2}(f))$ and the left hand side factor may be rewritten as

$$\mu_{\mathcal{H}}(t) = \sup_{I\in\mathcal{H}} \frac{|\sigma(I)|}{|I|}1_I(t), \qquad \text{where} \quad \mathcal{H} = \tau(\mathcal{L}),\ \sigma = \tau^{-1}.$$

Summing up, for any f with Haar support $\mathcal{L}$ we have the factorization

$$S(T_{\tau,1}(f))(t) \leq \mu_{\mathcal{H}}(t)^{1/2} S(T_{\tau,2}(f))(t) \quad \text{where} \quad \mathcal{H} = \tau(\mathcal{L}).$$

To $\tau : \mathcal{D} \to \mathcal{D}$ injective with inverse $\sigma : \tau(\mathcal{D}) \to \mathcal{D}$ we define S_σ to be the linear extension of the map $S_\sigma(h_I) = h_{\sigma(I)}$ when $I \in \tau(\mathcal{D})$ and $S_\sigma(h_I) = 0$ when $I \in \mathcal{D} \setminus \tau(\mathcal{D})$. The content of following proposition appeared in [14] by K. Smela. We present it here with a short proof emphasizing the connections of maximal functions to Carleson Measure and BMO. Only the right hand side of the inequality will be needed later in the extrapolation proof.

Proposition 1

$$\frac{\|S_\sigma\|^2_{BMO}}{A} \leq \sup_{\mathcal{H} \subseteq \tau(\mathcal{D})} \frac{1}{|\sigma(\mathcal{H})^*|} \int_0^1 \mu_{\mathcal{H}}(t) dt \leq \|S_\sigma\|^2_{BMO},$$

where $A > 0$ depends on the constants of the atomic decomposition for H^1.

Proof. First estimate the right hand side. Fix $\mathcal{H} \subseteq \mathcal{D}$. Without loss of generality we assume that $\mathcal{H}$ is a finite collection of intervals. In the first step of the argument we resolve the maximal function. To $t \in [0,1]$ choose $J_t \in \mathcal{H}$ so that

$$\frac{|\sigma(J_t)|}{|J_t|} = \sup_{J \in \mathcal{H}} \frac{|\sigma(J)|}{|J|} 1_J(t).$$

Put $\mathcal{B} = \{J_t : t \in [0,1]\} \subseteq \mathcal{H}$. Fubini's theorem yields

$$\int_0^1 \sup_{J \in \mathcal{H}} \frac{|\sigma(J)|}{|J|} 1_J(t) dt = \sum_{K \in \mathcal{B}} |\sigma(K)| \frac{|K| - |G_1^*(K|\mathcal{B})|}{|K|}. \tag{13}$$

Thus resolving the maximal functions led us to evaluating Carleson Measure. Next we obtain estimates for the right hand side of (13). For $K \in \mathcal{B}$ put

$$c_K^2 = (|K| - |G_1^*(K|\mathcal{B})|)/|K|$$

and define

$$f = \sum_{K \in \mathcal{B}} c_K h_K.$$

Observe that $\|f\|_{\mathrm{BMO}} = 1$. Indeed, for $I_0 \in \mathcal{B}$ write

$$\begin{aligned} \sum_{K \in \mathcal{B},\, K \subseteq I_0} |K| c_K^2 &= \sum_{K \in \mathcal{B},\, K \subseteq I_0} |K| - |G_1^*(K|\mathcal{B})| \\ &= |I_0|. \end{aligned}$$

Next observe that the right hand side of (13) coincides with $\|S_\sigma f\|_2^2$. We have $S_\sigma f = \sum_{K\in\mathcal{B}} c_K h_{\sigma(K)}$, and

$$\begin{aligned} \|\sum_{K\in\mathcal{B}} c_K h_{\sigma(K)}\|_2^2 &= \sum_{K\in\mathcal{B}} |\sigma(K)| c_K^2 \\ &= \sum_{K\in\mathcal{B}} |\sigma(K)| \frac{|K| - |G_1^*(K|\mathcal{B})|}{|K|}. \end{aligned} \tag{14}$$

The Haar support of $S_\sigma f$ is $\sigma(\mathcal{B})$ which is contained in $\sigma(\mathcal{H})$. Hence

$$\|S_\sigma f\|_2^2 \le |\sigma(\mathcal{H})^*| \|S_\sigma f\|_{\mathrm{BMO}}^2.$$

Combining (13) and (14) with the fact that $\|f\|_{\mathrm{BMO}} = 1$ we obtain

$$\begin{aligned} \int_0^1 \mu_{\mathcal{H}}(t)dt &\le |\sigma(\mathcal{H})^*| \|S_\sigma f\|_{\mathrm{BMO}}^2 \\ &\le |\sigma(\mathcal{H})^*| \|S_\sigma\|_{\mathrm{BMO}}^2. \end{aligned}$$

This proves the right hand side estimate.

Next we turn to the left hand side estimate using $H^1-\mathrm{BMO}$ duality. We prove that

$$\|T_{\tau,1}\|_1^2 \le A_1 C_1,$$

where A_1 is the constant appearing in the atomic decomposition for H^1 and

$$C_1 = \sup_{\mathcal{H}\subseteq\tau(\mathcal{D})} \frac{1}{|\sigma(\mathcal{H})^*|} \int_0^1 \mu_{\mathcal{H}}(t)dt \quad \text{with} \quad \mu_{\mathcal{H}}(t) = \sup_{J\in\mathcal{H}} \frac{|\sigma(J)|}{|J|} 1_J(t).$$

Fix $I \in \mathcal{D}$ and let $f : [0,1] \to \mathbb{R}$ be a dyadic H^1 atom so that

$$\operatorname{supp} S^2(f) \subseteq I \quad \text{and} \quad \|f\|_2 \le |I|^{-1/2}. \tag{15}$$

The square functions $S(T_{\tau,1}(f))$ and $S(T_{\tau,2}(f))$ are related by pointwise factorization,

$$S(T_{\tau,1}(f)) \le \mu_{\mathcal{H}}^{1/2} S(T_{\tau,2}(f))$$

where $\mathcal{H} = \tau(\{J \in \mathcal{D} : J \subseteq I_0\})$. Integrating and using the Cauchy Schwarz inequality gives,

$$\begin{aligned} \|T_{\tau,1}(f)\|_{H^1} &\le \left(\int_0^1 \mu_{\mathcal{H}}\right)^{1/2} \|T_{\tau,2}(f)\|_2 \\ &\le C_1^{1/2} |\sigma(\mathcal{H})^*|^{1/2} \|f\|_2. \end{aligned} \tag{16}$$

Since $\sigma(\mathcal{H})^* \subseteq I$ and f is a dyadic atom satisfying (15) we get from (16) that $\|T_{\tau,1}\|_{H^1} \leq (A_1 C_1)^{1/2}$ and by duality,

$$\|S_\sigma\|^2_{\mathrm{BMO}} \leq AC_1.$$

■

Remark. Compare the inequalities of Proposition 1 with (4). For

$$C_1 = \sup_{\mathcal{H}\subseteq\tau(\mathcal{D})} \frac{1}{|\sigma(\mathcal{H})^*|} \int_0^1 \mu_{\mathcal{H}}(t)dt \quad \text{with} \quad \mu_{\mathcal{H}}(t) = \sup_{J\in\mathcal{H}} \frac{|\sigma(J)|}{|J|} 1_J(t)$$

we get

$$C_1 \sim \sup \frac{[\![\sigma(\mathcal{C})]\!]}{[\![\mathcal{C}]\!]},$$

where the supremum is extended over all $\mathcal{C} \subseteq \tau(\mathcal{D})$.

Theorem 2 *Let $0 < p \leq 2$. Then for any $0 < q \leq p$,*

$$\|T_{\tau,q} \otimes \mathrm{Id}_X\|_q \leq A_q C_1^{1/q-1/p} \|T_{\tau,p} \otimes \mathrm{Id}_X\|_p,$$

where

$$C_1 = \sup_{\mathcal{H}\subseteq\tau(\mathcal{D})} \frac{1}{|\sigma(\mathcal{H})^*|} \int_0^1 \mu_{\mathcal{H}}(t)dt$$

and $A_q > 0$ is determined by the atomic decomposition for H^q_X.

Proof. Let $0 < q \leq p \leq 2$. Let f be an X valued H^q atom, so that

$$\operatorname{supp} \mathbb{S}(f) \subseteq I \quad \text{and} \quad \mathbb{S}(f) \leq |I|^{-1/q}.$$

Comparing the defining equations for $T_{\tau,p} \otimes \mathrm{Id}_X(f)$ and $T_{\tau,q} \otimes \mathrm{Id}_X(f)$, gives the pointwise estimate between square functions,

$$\mathbb{S}(T_{\tau,q} \otimes \mathrm{Id}_X(f))(t) \leq \sup_{I\in\mathcal{L}} \left[\frac{|I|}{|\tau(I)|} 1_{\tau(I)}(t) \right]^{1/q-1/p} \mathbb{S}(T_{\tau,p} \otimes \mathrm{Id}_X(f))(t).$$

Hence with $\mathcal{H} = \tau^{-1}(\{J \in \mathcal{D} : J \subseteq I\})$ we get the factorization

$$\mathbb{S}(T_{\tau,q} \otimes \mathrm{Id}_X(f)) \leq \mu_{\mathcal{H}}^{1/q-1/p} \mathbb{S}(T_{\tau,p} \otimes \mathrm{Id}_X(f)).$$

Next raise the above estimate to the power q and apply Hoelder's inequality with conjugate indices p/q and $p/(p-q)$. This gives

$$\int \mathbb{S}(T_{\tau,q} \otimes \text{ Id}_X(f))^q dt \leq \left(\int \mu_{\mathcal{H}}(t)dt\right)^{1-q/p} \left(\int \mathbb{S}(T_{\tau,p} \otimes \text{ Id}_X(f))^p dt\right)^{q/p}.$$

Taking $q-$th root yields

$$\|T_{\tau,q} \otimes \text{ Id}_X(f)\|_{H^q_X} \leq \left(\int \mu_{\mathcal{H}}(t)dt\right)^{1/q-1/p} \|T_{\tau,p} \otimes \text{ Id}_X(f)\|_{H^p_X}.$$

Since $\int \mu_{\mathcal{H}} \leq C_1|I|$ and $\|f\|_{H^p_X} \leq |I|^{1/p-1/q}$, we get

$$\|T_{\tau,q} \otimes \text{ Id}_X(f)\|_{H^q_X} \leq C_1^{1/q-1/p} \|T_{\tau,p} \otimes \text{ Id}_X\|_p.$$

The atomic decomposition theorem for H^q_X implies now

$$\|T_{\tau,q} \otimes \text{ Id}_X\|_q \leq A_q C_1^{1/q-1/p} \|T_{\tau,p} \otimes \text{ Id}_X\|_p.$$

■

In the case when $p > 0$ is strictly less that 2, the conclusion of the previous theorem can be turned into concise extrapolation estimates as follows.

Theorem 3 *Let* $0 < q \leq p < 2$. *Then*

$$\|T_{\tau,q} \otimes \text{ Id}_X\|_q^{\frac{q}{2-q}} \leq A(p,q) \|T_{\tau,p} \otimes \text{ Id}_X\|_p^{\frac{p}{2-p}}.$$

Proof. Let $0 < p < 2$ (and p strictly less than 2). By (3) we have the scalar valued extrapolation estimate [5]

$$\|T_{\tau,1}\|_1 \leq A(p) \|T_{\tau,p}\|_p^{\frac{p}{2-p}}.$$

Recall that $C_1 \leq (A\|T_{\tau,1}\|_1)^2$. Hence the above inequality gives

$$C_1^{\frac{p-q}{pq}} \leq (A\|T_{\tau,p}\|_p)^{\frac{2(p-q)}{p(2-p)}}.$$

Next note that $\frac{2(p-q)}{p(2-p)} + 1 = \frac{p(2-q)}{q(2-p)}$. It remains to invoke Theorem 2 to obtain

$$\begin{aligned}\|T_{\tau,q} \otimes \text{ Id}_X\|_q &\leq A(p,q) C_1^{\frac{p-q}{pq}} \|T_{\tau,p} \otimes \text{ Id}_X\|_p \\ &\leq A(p,q) \|T_{\tau,p} \otimes I_X\|_p^{\frac{p(2-q)}{q(2-p)}}.\end{aligned}$$

as claimed.

■

Remark. Our proof identifies the separate contribution of $\|T_{\tau,p}\|_p$ and $\|T_{\tau,p} \otimes \ \mathrm{Id}_X\|_p$ to the upper bound for $\|T_{\tau,q} \otimes \ \mathrm{Id}_X\|_q$. It gives

$$\|T_{\tau,q} \otimes \ \mathrm{Id}_X\|_q \leq A(p,q)\|T_{\tau,p}\|_p^{2\left(\frac{1}{q}-\frac{1}{p}\right)}\|T_{\tau,p} \otimes \ \mathrm{Id}_X\|_p. \tag{17}$$

The extrapolation estimates of Theorem 2 and Theorem 3 hold for one fixed Banach space X. Now we change the nature of our assumptions and demand boundedness of the vector valued rearrangement operator for *each* Banach space with the UMD property. While this formulates a more restrictive hypothesis the resulting conclusion is also stronger. The next theorem is a consequence of Proposition 5 below and of the extrapolation theorems in [4]. We point out that in the hypothesis of Theorem 4 the value $p_0 = 2$ is included.

Theorem 4 *If there exists* $1 < p_0 < \infty$ *so that*

$$\|T_{\tau,p_0} \otimes \ \mathrm{Id}_E\|_{L_E^{p_0}} < \infty, \quad \textit{for each UMD space } E. \tag{18}$$

then $\|T_{\tau,1}^{-1}\|_1 < \infty$ *and for any* $1 < p < \infty$,

$$\|T_{\tau,p} \otimes \ \mathrm{Id}_E\|_{L_E^p}\|T_{\tau,p}^{-1} \otimes \ \mathrm{Id}_E\|_{L_E^p} < \infty,$$

for each UMD space E.

Consider a rearrangement operator (of the H^1 normalized Haar system) that is unbounded on H^1. Then as shown in [11], for any n, there exist vectors $x_1, \ldots, x_n$ in H^1 that are equivalent to the unit vector basis of ℓ_n^2, so that their images are equivalent to the unit vector basis of ℓ_n^1. Guided by the reasoning of [3, Example 3.2] we next give the vector valued interpretation of Proposition 2 and Lemma 3 +4 in [11]. Thus we show that (18) implies $\|T_{\tau,1}^{-1}\|_1 < \infty$.

Proposition 5 *For any Banach space* Y *and* $1 < p \leq 2$ *we have*

$$\|T_{\tau,1}^{-1}\|_1 = \infty \quad \textit{implies} \quad \mathit{Type}_p(Y) \leq C\|T_{\tau,p} \otimes \ \mathrm{Id}_Y\|_{L_Y^{p_0}}.$$

Consequently, if

$$\|T_{\tau,p} \otimes \ \mathrm{Id}_E\|_{L_E^p} < \infty, \quad \textit{for each UMD space } E,$$

then

$$\|T_{\tau,1}^{-1}\|_1 < \infty.$$

PROOF. Suppose that $\|T_{\tau,1}^{-1}\|_1 = \infty$. By [11] (see also Proposition 3.3.2 and Theorem 3.3.5 in [12]) for each $n \in \mathbb{N}$ there exists a collection of dyadic intervals $\mathcal{C}$ so that

$$[\![\mathcal{C}]\!] \leq 4 \quad \text{and} \quad [\![\tau(\mathcal{C})]\!] \geq n^2.$$

By the Carleson-Garnett condensation lemma, (see [12, Lemma 3.1.4]), there exists $K \in \tau(\mathcal{C})$ so that

$$|G_n(K|\tau(\mathcal{C}))^*| \geq (1 - \frac{1}{n})|K|, \tag{19}$$

where $G_n(K|\tau(\mathcal{C}))$ denotes the $n-$th generation of $\tau(\mathcal{C})$ that is underneath K. By rescaling we may assume that $K = [0,1]$. Now for $i \leq n$, put

$$\mathcal{F}_i = G_i(K|\tau(\mathcal{C})) \quad \text{and} \quad \mathcal{E}_i = \tau^{-1}G_i(K|\tau(\mathcal{C}))$$

to define

$$\rho_i = \sum_{\tau(J)\in\mathcal{F}_i} h_{\tau(J)} \quad \text{and} \quad s_i = \sum_{J\in\mathcal{E}_i} |\tau(J)|^{1/p}\frac{h_J}{|J|^{1/p}}h_J.$$

Note that

$$\bigcup_{i=1}^{n}\mathcal{E}_i \subseteq \mathcal{C} \quad \text{and} \quad T_{\tau,p}s_i = \rho_i \quad i \leq n.$$

Since $[\![\mathcal{C}]\!] \leq 4$ we have with [4, Lemma 3] that for $a_i \in Y$

$$\begin{aligned} \|\sum_{i=1}^{n} a_i s_i\|_{L_Y^p} &\leq C\left(\sum_{i=1}^{n}\|a_i\|_Y^p \sum_{J\in\mathcal{E}_i}|\tau(J)|\right)^{1/p} \\ &\leq C\left(\sum_{i=1}^{n}\|a_i\|_Y^p\right)^{1/p}. \end{aligned} \tag{20}$$

Now let $\{r_i\}_{i=1}^n$ be the first n of the independent $\{\pm 1\}$ valued Rademacher functions. It follows from (19) that for $a_i \in Y$

$$\|\sum_{i=1}^{n} a_i r_i\|_{L_Y^p} \leq \|\sum_{i=1}^{n} a_i \rho_i\|_{L_Y^p} + \frac{1}{n}\left(\sum_{i=1}^{n}\|a_i\|_Y\right). \tag{21}$$

Hence for $n \in \mathbb{N}$ large enough, by (20) and (21), we get with $\rho_i = T_{\tau,p} s_i$

$$\begin{aligned}
\| \sum_{i=1}^{n} a_i r_i \|_{L^p_Y} &\le \| \sum_{i=1}^{n} a_i T_{\tau,p} s_i \|_{L^p_Y} + \frac{1}{n} \left(\sum_{i=1}^{n} \|a_i\|_Y \right) \\
&\le C \|T_{\tau,p} \otimes \ \mathrm{Id}_Y\|_{L^p_Y} \| \sum_{i=1}^{n} a_i s_i \|_{L^p_Y} + \frac{1}{n} \left(\sum_{i=1}^{n} \|a_i\|_Y \right) \qquad (22) \\
&\le C \|T_{\tau,p} \otimes \ \mathrm{Id}_Y\|_{L^p_Y} \left(\sum_{i=1}^{n} \|a_i\|_Y^p \right)^{1/p} .
\end{aligned}$$

Since $i \le n$ and $a_i \in Y$ were chosen arbitrary (22) implies

$$\mathrm{Type}_p(Y) \le C \|T_{\tau,p} \otimes \ \mathrm{Id}_Y\|_{L^p_Y}.$$

To see the moreover part of Proposition 5 just test the above estimate with $Y = L^r$, and $1 < r < p$.

■

Proof of Theorem 4. It suffices to consider $1 < p_0 \le 2$. Consider first the case $p_0 = 2$. Since UMD spaces are reflexive, and the UMD property is a self dual isomorphic invariant we get from (18) by dualization that

$$\|T_{\tau,2} \otimes \ \mathrm{Id}_E\|_{L^2_E} \|T^{-1}_{\tau,2} \otimes \ \mathrm{Id}_E\|_{L^2_E} < \infty, \tag{23}$$

for any UMD space E. Hence by Proposition 5

$$\|T^{-1}_{\tau,1}\|_1 \|T_{\tau,1}\|_1 < \infty. \tag{24}$$

By [3, Corollary 5.6.], we get from (23) and (24) that for any $1 < p \le 2$ and any UMD space E

$$\|T_{\tau,p} \otimes \ \mathrm{Id}_E\|_{L^p_E} \|T^{-1}_{\tau,p} \otimes \ \mathrm{Id}_E\|_{L^p_E} < \infty.$$

By duality this gives the conclusion of Theorem 4 in the case $p_0 = 2$.

Next we turn to $p_0 < 2$. By scalar valued extrapolation we get then $\|T_{\tau,1}\|_1 < \infty$, and Proposition 5 yields $\|T^{-1}_{\tau,1}\|_1 < \infty$. Hence as in the previous case

$$\|T^{-1}_{\tau,1}\|_1 \|T_{\tau,1}\|_1 < \infty. \tag{25}$$

Moreover by reflexivity, duality, and the fact that UMD is a self dual isomorphic invariant, we obtain with $1/p_0 + 1/q_0 = 1$, that

$$\|T_{\tau,q_0}^{-1} \otimes \mathrm{Id}_E\|_{L_E^{q_0}} < \infty \quad \text{for each UMD space E.} \tag{26}$$

By [3, Corollary 5.6.] it follows from (25) and (26) that for each $q \le q_0$, the operator $T_{\tau,q}^{-1} \otimes \mathrm{Id}_E$ is bounded on L_E^q. Since $p_0 < 2$ and $q_0 > 2$, this holds in particular for $q = p_0$. In summary we have

$$\|T_{\tau,p_0} \otimes \mathrm{Id}_E\|_{L_E^{p_0}} \|T_{\tau,p_0}^{-1} \otimes \mathrm{Id}_E\|_{L_E^{p_0}} < \infty.$$

By [3, Corollary 5.8.] we obtain the conclusion of Theorem 4 in the case $p_0 < 2$.

■

References

[1] O. Blasco and Q. H. Xu. Interpolation between vector-valued Hardy spaces. *J. Funct. Anal.*, 102(2):331–359, 1991.

[2] D. L. Burkholder. Distribution function inequalities for martingales. *Ann. Probability*, 1:19–42, 1973.

[3] S. Geiss and P. F. X. Müller. Extrapolation of vector valued rearrangement operators. *www.math.jyu.fi/research/pspdf/350.pdf*.

[4] S. Geiss and P. F. X. Müller. Haar type and Carleson constants. *Bull. Lond. Math. Soc.*, 40(3):432–438, 2008.

[5] S. Geiss, P. F. X. Müller, and V. Pillwein. A remark on extrapolation of rearrangement operators on dyadic H^s, $0 < s \le 1$. *Studia Math.*, 171(2):197–205, 2005.

[6] S. Janson and P. W. Jones. Interpolation between H^p spaces: the complex method. *J. Funct. Anal.*, 48(1):58–80, 1982.

[7] J.-P. Kahane. *Some random series of functions*, volume 5 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1985.

[8] B. Maurey. *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p.* Société Mathématique de France, Paris, 1974. With an English summary, Astérisque, No. 11.

[9] B. Maurey. Quelques problèmes de factorisation d'opérateurs linéaires. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 2*, pages 75–79. Canad. Math. Congress, Montreal, Que., 1975.

[10] B. Maurey. Système de Haar. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. I et II*, pages 26 pp. (erratum, p. 1). Centre Math., École Polytech., Paris, 1975.

[11] P. F. X. Müller. Rearrangements of the Haar system that preserve BMO. *Proc. London Mth. Soc. (3)*, 75(3):600–618, 1997.

[12] P. F. X. Müller. *Isomorphisms between H^1 spaces*, volume 66 of *Mathematical Monographs (New Series)]*. Birkhäuser Verlag, Basel, 2005.

[13] P. F. X. Müller and G. Schechtman. Several results concerning unconditionality in vector valued L^p and $H^1(\mathcal{F}_n)$ spaces. *Illinois J. Math.*, 35(2):220–233, 1991.

[14] K. Smela. Continuous rearrangements of the haar system in H_p. *Studia Math.*, 189:189–199, 2008.

[15] P. Wojtaszczyk. *Banach spaces for analysts*, volume 25 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991.

Address: Institut für Analysis
J. Kepler Universität
A4040 Linz Austria
pfxm@bayou.uni-linz.ac.at